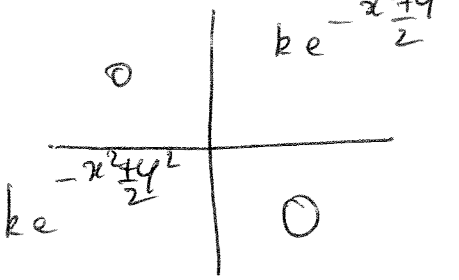
TD4 - Probabilites

Exercice 1



1) $$\iint_{(\mathbb{R}^+)^2 \cup (\mathbb{R}^-)^2} k e^{-\frac{x^2+y^2}{2}} \, dx dy = 1$$

Par symetrie $\displaystyle\iint_{(\mathbb{R}^+)^2} e^{-\frac{1}{2}(x^2+y^2)} \, dx dy = \iint_{(\mathbb{R}^-)^2} e^{-\frac{1}{2}(x^2+y^2)} \, dx dy$

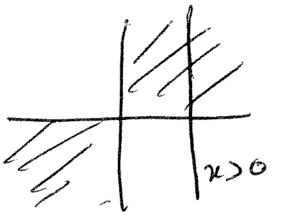
(il suffit de faire le changement de variable $\begin{matrix} x' = -x \\ y' = -y \end{matrix}$ dans l'une des 2 intégrals)

De plus $$\iint_{(\mathbb{R}^+)^2} e^{-\frac{1}{2}(x^2+y^2)} \, dx dy = \underbrace{\left(\int_0^{+\infty} e^{-\frac{1}{2}x^2} dx\right)}_{\frac{\sqrt{2\pi}}{2}} \underbrace{\left(\int_0^{+\infty} e^{-\frac{1}{2}y^2} dy\right)}_{\frac{\sqrt{2\pi}}{2}}$$

$$= \frac{2\pi}{4} = \frac{\pi}{2}$$

donc $2k\left(\frac{\pi}{2}\right) = 1 \Rightarrow \boxed{k = \frac{1}{\pi}}$

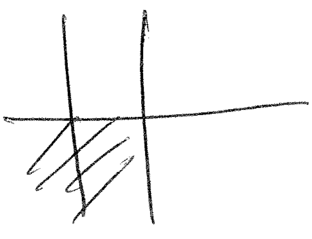
2)



si $x>0$ $$f(x,\cdot) = \int_0^{+\infty} \frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} dy$$

$$= \frac{1}{\pi} e^{-\frac{x^2}{2}} \underbrace{\int_0^{+\infty} e^{-y^2/2} dy}_{\frac{\sqrt{2\pi}}{2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



si $x<0$ $$f(x,\cdot) = \int_{-\infty}^{0} \frac{1}{\pi} e^{-\frac{x^2+y^2}{2}} dy$$

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

donc $f(x,\cdot) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ $\forall x \in \mathbb{R}$, i.e., $\boxed{X \sim \mathcal{N}(0,1)}$

Par symétrie $\boxed{Y \sim N(0,1)}$

3) $Cov(X,Y) = E[XY] - \underbrace{E[X]}_{0 \text{ car } X \sim N(0,1)} E[Y]$

$$= E[XY]$$

$$= \iint xy\, f(x,y)\, dx\,dy = 2 \int_0^{+\infty}\int_0^{+\infty} \frac{xy}{\pi} e^{-\frac{1}{2}\frac{x^2+y^2}{2}}\, dx\,dy$$

$$cov(X,Y) = \frac{2}{\pi}\left(\underbrace{\int_0^{+\infty} x e^{-\frac{1}{2}x^2} dx}_{\left[-e^{-\frac{x^2}{2}}\right]_0^{+\infty} = 1}\right)\left(\underbrace{\int_0^{+\infty} y e^{-y^2/2} dy}_{1}\right)$$

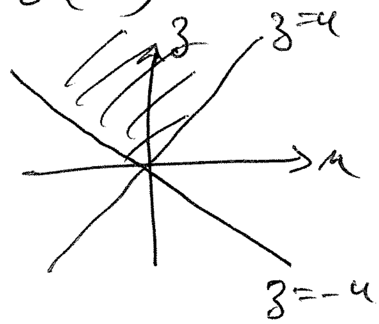
d'où $\boxed{cov(X,Y) = \frac{2}{\pi}}$

4) $\begin{cases} Z = X+Y \\ U = X-Y \end{cases} \iff \begin{cases} X = \frac{1}{2}(Z+U) \\ Y = \frac{1}{2}(Z-U) \end{cases}$

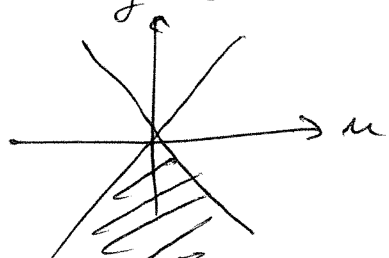
Donc le changement de variables est bijectif de $(\mathbb{R}^+)^2 \cup (\mathbb{R}^-)^2 \times \Delta$

$\boxed{\Delta ?}$

$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases} \iff \begin{cases} z+u \geq 0 \\ z-u \geq 0 \end{cases} \iff \begin{cases} z \geq u \\ \text{et} \\ z \geq -u \end{cases}$



$\begin{cases} x \leq 0 \\ y \leq 0 \end{cases} \iff \begin{cases} z \leq u \\ z \leq -u \end{cases}$



donc Δ est la réunion des deux domaines hachurés ci-dessus

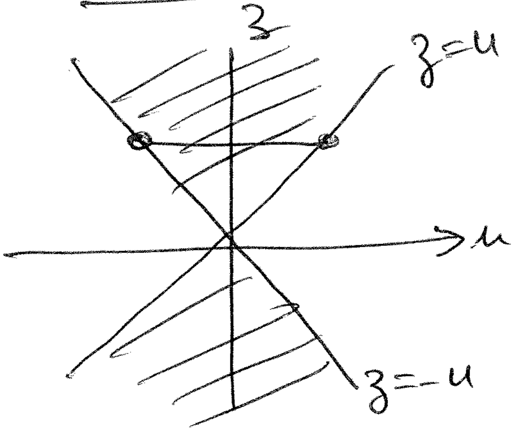
$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \quad \text{donc } |J| = \frac{1}{2}$$

La densité de ~~z,u~~ (Z,U) est

$$g(z,u) = \frac{1}{\pi} \exp\left\{-\frac{1}{2}\left[\frac{(z+u)^2}{4} + \frac{(z-u)^2}{4}\right]\right\} \times \frac{1}{2}$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{1}{8}\left[z^2+u^2+\cancel{2zu}+z^2+u^2-\cancel{2zu}\right]\right\}$$

$$\boxed{g(z,u) = \frac{1}{2\pi} \exp\left\{-\frac{z^2+u^2}{4}\right\} \quad (z,u)\in D}$$



loi de Z pour $z>0$ on voit que $u\in[-z,+z]$

donc $$g(z,\cdot) = \int_{-z}^{+z} \frac{1}{2\pi} e^{-z^2/4} e^{-u^2/4}\, du$$

$$= \int_{-\frac{z}{\sqrt{2}}}^{z/\sqrt{2}} \frac{1}{\sqrt{\cancel{2}\pi}} e^{-z^2/4} e^{-v^2/2} \frac{\cancel{\sqrt{2}}\, dV}{\sqrt{2\pi}}$$

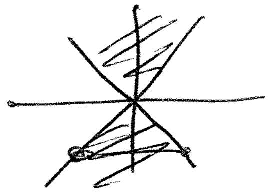
on pose $\frac{u^2}{4} = \frac{v^2}{2}$

$v = \frac{u}{\sqrt{2}}$

$$= \frac{1}{\sqrt{\pi}} e^{-z^2/4} \underbrace{\int_{-z/\sqrt{2}}^{z/\sqrt{2}} \frac{1}{\sqrt{2\pi}} e^{-v^2/2}\, dV}_{\Phi(z/\sqrt{2}) - \Phi(-z/\sqrt{2})}$$

donc pour $\boxed{z>0 \quad g(z,\cdot)=\frac{1}{\sqrt{\pi}}e^{-z^2/4}\left[\phi(z/\sqrt{2})-\phi(-z/\sqrt{2})\right]}$

Pour $z<0$ $\quad u\in[z,-z]$ donc il suffit de changer z en $-z$



Finalement $\boxed{g(z,\cdot)=\frac{1}{\sqrt{\pi}}e^{-z^2/4}\left[\phi\left(\frac{|z|}{\sqrt{2}}\right)-\phi\left(-\frac{|z|}{\sqrt{2}}\right)\right] \quad \forall z\in\mathbb{R}}$

Idem pour U

Ex2 et 3 : corrections sur ma page web